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Anomalous exponents to order ε^3 in the rapid-change model of passive scalar advection

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Field-theoretic renormalization group is applied to the Kraichnan model of a passive scalar advected by the Gaussian velocity field with the covariance $\langle \mathbf{v}(t,\mathbf{x})\mathbf{v}(t',\mathbf{x})\rangle - \langle \mathbf{v}(t,\mathbf{x})\mathbf{v}(t',\mathbf{x}')\rangle \propto \delta(t-t')|\mathbf{x}-\mathbf{x}'|^{\varepsilon}$. Inertial-range anomalous exponents, related to the scaling dimensions of tensor composite operators built of the scalar gradients, are calculated to the order ε^3 of the ε expansion. The nature and the convergence of the ε expansion in the models of turbulence are briefly discussed.

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The investigation of intermittency and anomalous scaling in fully developed turbulence remains essentially an open theoretical problem. Both the natural and numerical experiments suggest that the deviation from the predictions of the classical Kolmogorov theory [1] is even more strongly pronounced for a passively advected scalar field than for the velocity field itself; see, e.g., [2] and literature cited therein. At the same time, the problem of passive advection appears to be easier tractable theoretically: even simplified models describing the advection by a ''synthetic'' velocity field with a given Gaussian statistics reproduce many of the anomalous features of genuine turbulent heat or mass transport observed in experiments. Therefore, the problem of passive scalar advection, being of practical importance in itself, may also be viewed as a starting point in studying anomalous scaling in the turbulence on the whole.

Most progress has been achieved for the so-called rapid-change model [3]: the anomalous exponents have been calculated on the basis of a microscopic model and within regular perturbation expansions; see Refs. [3–15] and references therein.

In that model, the advection of a passive scalar field $\theta(x)\equiv\theta(t,\mathbf{x})$ is described by the stochastic equation

$$\partial_t\theta+(v_i\partial_i)\theta=\nu_0\Delta\theta+f, \tag{1}$$

where $\partial_t\equiv\partial/\partial t$, $\partial_i\equiv\partial/\partial x_i$, ν_0 is the molecular diffusivity coefficient, Δ is the Laplace operator, $\mathbf{v}(x)\equiv\{v_i(x)\}$ is the transverse (owing to the incompressibility) velocity field, and $f\equiv f(x)$ is an artificial Gaussian scalar noise with zero average and correlator

$$\langle f(x)f(x')\rangle=\delta(t-t')C(r/L),\quad r=|\mathbf{x}-\mathbf{x}'|. \tag{2}$$

The parameter L is an integral scale related to the scalar noise, and $C(r/L)$ is some function finite as $L\to\infty$. Without loss of generality, we take $L=\infty$ and $C(0)=1$.

In the real problem, the field $\mathbf{v}(x)$ satisfies the Navier-Stokes equation. In the rapid-change model it obeys a Gaussian distribution with zero mean and correlator

$$\langle v_i(x)v_j(x')\rangle=D_0\frac{\delta(t-t')}{(2\pi)^d}\int d\mathbf{k}\,P_{ij}(\mathbf{k})k^{-d-\varepsilon}\times\exp[i\mathbf{k}\cdot(\mathbf{x}-\mathbf{x}')], \tag{3}$$

where $P_{ij}(\mathbf{k})=\delta_{ij}-k_ik_j/k^2$ is the transverse projector, $k\equiv|\mathbf{k}|$, $D_0>0$ is an amplitude factor, d is the dimensionality of the $\mathbf{x}$ space, and $0<\varepsilon<2$ is a parameter with the real (''Kolmogorov'') value $\varepsilon=4/3$. The infrared (IR) regularization is provided by the cut-off in integral (3) from below at $k\simeq m$, where $m\equiv 1/\ell$ is the reciprocal of another integral scale ℓ; the precise form of the cut-off is not essential. The relation $D_0/\nu_0=\Lambda^{\varepsilon}$ defines the characteristic ultraviolet momentum scale Λ.

The issue of interest is, in particular, the behavior of the equal-time structure functions

$$S_n(r)=\langle[\theta(t,\mathbf{x})-\theta(t,\mathbf{x}')]^n\rangle,\quad r=|\mathbf{x}-\mathbf{x}'| \tag{4}$$

in the inertial-convective range $\Lambda\gg 1/r\gg m$.

In the isotropic model (1)–(3), the odd multipoint correlation functions of the scalar field vanish, while the even equal-time functions satisfy linear partial differential equations [3–5]. The solution for the pair correlator is obtained explicitly; it shows that the structure function S_2 is finite for $m=0$ [3]. The higher-order correlators are not found explicitly, but their asymptotic behavior for $m\to 0$ can be extracted from the analysis of the nontrivial zero modes of the corresponding differential operators in the limits $1/d\to 0$ [4], $\varepsilon\to 0$ [5,7], or $\varepsilon\to 2$ [6,7]. It was shown that the structure functions in the inertial-convective range exhibit anomalous scaling behavior:

$$S_{2n}(r)\propto D_0^{-n}r^{n(2-\varepsilon)}(mr)^{\Delta_n} \tag{5}$$

with negative anomalous exponents Δ_n, whose first terms of the expansion in $1/d$ [4] and ε [5] have the forms

$$\begin{aligned}\Delta_n&=-n(n-2)\varepsilon/2d+O(1/d^2)\\&=-n(n-2)\varepsilon/2(d+2)+O(\varepsilon^2).\end{aligned} \tag{6}$$

In paper [9], the field-theoretic renormalization group (RG) and operator product expansion (OPE) were applied to model (1)–(3). In the RG approach, the anomalous scaling for the structure functions and various pair correlators is established as a consequence of the existence in the corresponding operator product expansions of ''dangerous'' composite operators (powers of the local dissipation rate), whose *negative* critical dimensions determine the anomalous exponents Δ_n. The anomalous exponents were calculated in Ref.

[9] to the order ε^2 of the ε expansion for the arbitrary value of d; generalization to the compressible case was given in [10,11]. The main advantage of the RG approach (apart from its calculational efficiency) is the universality: it can equally be applied to the case of finite correlation time [12].

In this Rapid Communication, we present the anomalous exponents and other quantities for model (1)–(3) in the order ε^3. Here we give only basic ideas and results; more exhaustive discussion of the calculational technique will be given elsewhere. A general review of the RG approach to the statistical theory of turbulence can be found in Refs. [16,17]; the case of the Kraichnan model is discussed in [9] in detail.

The stochastic problem (1)–(3) can be reformulated as a multiplicatively renormalizable field-theoretic model; the corresponding RG equations have an IR attractive fixed point. This implies existence of the infrared scaling behavior for all correlation functions with certain scaling dimensions, calculated as series in ε (in this sense, the exponent ε plays in the RG approach the same part as the parameter $\varepsilon=4-d$ does in the RG theory of critical behavior). In particular, for the structure functions (4) and (5) in the IR asymptotic range ($\Lambda r \gg 1$) one obtains

$$S_{2n}(r) \propto D_0^{-n} r^{n(2-\varepsilon)} \chi_n(mr). \tag{7}$$

The behavior of the scaling functions $\chi_n(mr)$ at $mr \to 0$ (inertial-convective range) is obtained with the aid of the operator product expansion:

$$\chi_n(mr) = \sum_F C_F (mr)^{\Delta_F}, \tag{8}$$

where the sum runs over all possible composite operators F entering the OPE for a given structure function, Δ_F are their critical dimensions, and C_F are numerical coefficients analytical in $(mr)^2$ and finite at $mr=0$.

The key role is played by the critical dimensions Δ_{nl}, associated with the tensor composite operators

$$F_{nl} = \partial_{i_1}\theta \cdots \partial_{i_l}\theta (\partial_i \theta \partial_i \theta)^p, \tag{9}$$

where l is the number of the free vector indices and $n=l+2p$ is the total number of the fields θ entering the operator; the vector indices of the symbol F_{nl} are omitted.

The dimension $\Delta_n \equiv \Delta_{n0}$ of the scalar operator is nothing other than the anomalous exponent in Eq. (5); see Ref. [9]. The dimensions with $l \neq 0$ become relevant if the forcing (2) is anisotropic: Δ_{nl} corresponds to the zero-mode contribution to the lth term of the Legendre decomposition for the function S_n; see Ref. [12]. They can be systematically calculated as series in ε,

$$\Delta_{nl} = \sum_{k=1}^{\infty} \Delta_{nl}^{(k)} \varepsilon^k, \tag{10}$$

with the first-order coefficient [12]

$$\Delta_{nl}^{(1)} = -\frac{n(n-2)}{2(d+2)} + \frac{(d+1)l(d+l-2)}{2(d-1)(d+2)} \tag{11}$$

(see also Refs. [13,14]; for $l=0$ this gives the result of [5], while for $n=3$ and $l=1$ or 3 the result of [7] is recovered).

The coefficients $\Delta_{n0}^{(2)}$ and $\Delta_{n2}^{(2)}$ were obtained in Ref. [9] for any n and d; the result for general l is presented in [11]. In particular, one has

$$\Delta_{nl}^{(2)} = n(n-2)(0.000203n - 0.02976) - l^2(0.01732n + 0.01223) \tag{12}$$

for $d=2$ and

$$\Delta_{nl}^{(2)} = n(n-2)(0.00203n - 0.00384) - l(l+1)(0.00710n - 0.00619) \tag{13}$$

for $d=3$ (analytical results are too cumbersome and will not be given here; see Refs. [9,11]).

Now let us turn to the $O(\varepsilon^3)$ contribution. No analytical result for it is available for general d; the numerical results have the forms

$$\Delta_{nl}^{(3)} = n(n-2)(0.005472n^2 + 0.0649n + 0.0647) + l^2(-0.02161n^2 - 0.1023n + 0.2406 + 0.01841 l^2) \tag{14}$$

for d=2 and

$$\Delta_{nl}^{(3)} = n(n-2)(0.00140n^2 + 0.0199n + 0.0343) + l(l+1) \times [-0.00420n^2 + 0.0241n + 0.0028(l^2 + l + 18)] \tag{15}$$

for $d=3$. The quantity $\Delta_{nl}^{(3)}$ can be expanded as a series in $1/d$; the coefficients of such an expansion can be found, in principle, to any given order. For general n and l to the order $1/d^2$ we have obtained

$$\Delta_{nl} = \varepsilon[-n(n-2)(1-2/d)/2d + (l/2)(1-2/d + l/d + 2/d^2)] + 3\varepsilon^2(n-2)(n-l)/4d^2 + \varepsilon^3(n-l) \times [1.74988(n-2) - 0.624916 l]/d^2. \tag{16}$$

Note that the ε^2 and ε^3 contributions decay for $d \to \infty$ faster than $1/d$ in agreement with the $O(1/d)$ result obtained in Ref. [4] for Δ_{n0}. Moreover, from Eq. (16) it follows that the leading $O(1/d^2)$ terms in these contributions vanish for $n=l$, so that the decay at $d \to \infty$ becomes even faster,

$$\Delta_{nn} = \varepsilon n/2 + n(n-1)\{\varepsilon/(d-1)(d+2) - \varepsilon^2[1 + (2n-7)/d]/d^3 - \varepsilon^3(3n-8)/2d^4\} + O(\varepsilon^4), \tag{17}$$

with the accuracy of $O(1/d^4)$.

We also recall that $\Delta_{20}=0$ to all orders in ε in agreement with the exact solution for the second-order structure function [3], and that the exact nonperturbative result for Δ_{22} exists for all ε and d [4].

For isotropic model (1)–(3), only scalar operators enter expansion (8), the number of the fields θ in the operators does not exceed the number of θ's on the left-hand side, and the leading term of the small-mr behavior is given by the operator with minimal dimension Δ_F. This allows one to

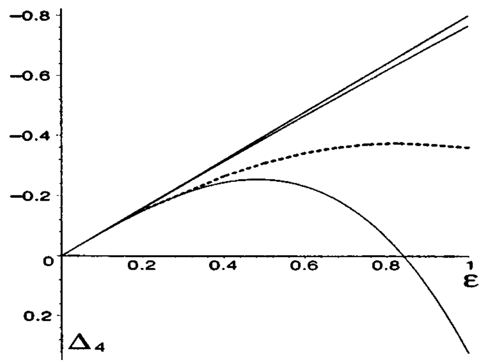

FIG. 1. The dimension Δ_4 for $d=3$ vs ε: the $O(\varepsilon)$, $O(\varepsilon^2)$, and $O(\varepsilon^3)$ approximations (from above to below). Dashed line: numerical simulation by Refs. [8].

identify the anomalous exponent Δ_n in Eqs. (5) and (6) with the critical dimension Δ_{n0} of the scalar operator F_{n0}.

If noise covariance (2) involves some fixed constant vector $\mathbf{n}$ (large-scale anisotropy), the above results for the dimensions Δ_{nl} do not change, but the operators with $l\neq 0$ also enter the right-hand side of Eq. (8) and give rise to contributions proportional to $P_l(z)$, the lth order Legendre polynomial, z being the angle between the vectors $\mathbf{n}$ and $\mathbf{r}$. The odd structure functions S_{2n+1} become nontrivial, and the leading term of their inertial-range behavior is determined by the dimension $\Delta_{2n+1,1}$ of the vector operator $F_{2n+1,1}$

$$S_{2n+1}(r)\propto D_0^{-n-1/2}r^{(n+1/2)(2-\varepsilon)}(mr)^{\Delta_{2n+1,1}} \tag{18}$$

(for more detail, see Refs. [12]).

In Fig. 1, we show the dimension Δ_4 (which determines the anomalous exponent for S_4) for $d=3$ in the first, second, and third orders in ε. In Figs. 2 and 3, we show the ''anomaly'' γ, defined by the relation $S_3\propto r^{3-\gamma}$, for $d=3$ and 2, respectively; note that the $O(\varepsilon^2)$ curve lies above the $O(\varepsilon)$ line for $n=3$ andn below it for $n=4$. In the same figures, we also present nonperturbative results obtained for $n=4$ in Refs. [8] using numerical simulations, and for $n=3$ in [7] using numerical integration of the zero mode equations ($\Delta_4=\zeta_4-2\zeta_2$ in the notation of [8] and $\gamma=3-\lambda$ in the notation of [7]).

An important issue which can be discussed on the example of the rapid-change model is that of the nature and convergence properties of ε expansions in models of turbulence and the possibility of their extrapolation to finite values $\varepsilon\sim 1$. Figures 1–3 show that the agreement between the ε expansion and nonperturbative results for small ε improves when the higher-order terms are taken into account, but the deviation becomes remarkable for $\varepsilon\sim 1$ and decreasing d. Furthermore, the coefficients of the ε series appear more irregular for $d=2$ (see Fig. 3), while the forms of the nonperturbative results [7,8] are not much affected by the choice of d.

Such behavior can be understood on the basis of the exact analytical result for Δ_{22}, which can be written in the form [4]

$$2\Delta_{22}=-d-2+\varepsilon+\sqrt{(\varepsilon+\varepsilon_+)(\varepsilon+\varepsilon_-)}, \tag{19}$$

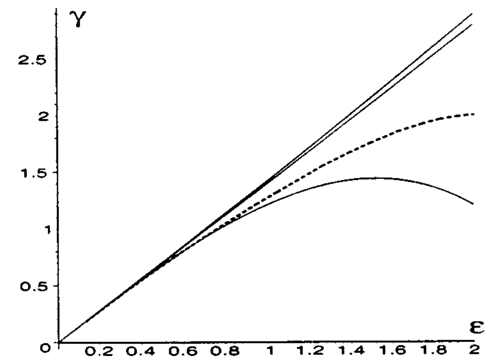

FIG. 2. The exponent γ for $d=3$ vs ε: the $O(\varepsilon^2)$, $O(\varepsilon)$, and $O(\varepsilon^3)$ approximations (from above to below). Dashed line: numerical solution by Refs. [7].

where

$$\varepsilon_\pm=[d^2+d+2\pm\sqrt{8d(d+1)}]/(d-1).$$

It shows that the corresponding ε expansion has the finite radius of convergence ε_-, ranging from 0 to ∞ when d varies from 1 to ∞; in particular, $\varepsilon_-\simeq 1.1$ for $d=2$ and $\varepsilon_-\simeq 2.1$ for $d=3$. Hence, the naive summation of the ε expansion for Δ_{22} works only in the interval $\varepsilon<\varepsilon_-$, which decreases almost linearly with $(d-1)$. In order to recover the behavior of Δ_{22} from its ε series for larger ε, it is necessary to isolate explicitly the singularity at ε_- in Eq. (19), thus changing to a kind of improved ε expansion (whose radius of convergence becomes $\varepsilon_+\gg\varepsilon_-$). In practice, the first three terms of this improved expansion approximate the exact result (19) equally well for all $0<\varepsilon<2$, both in two and three dimensions.

The difference with the models of critical phenomena, where ε series are always asymptotical, can be traced back to the fact that in the rapid-change models, there is no factorial growth of the number of diagrams in higher orders of the perturbation theory. The divergence for $d\to 1$ is naturally explained by the fact that the transverse vector field does not exist in one dimension (we also recall that the RG fixed point diverges at $d=1$; see Ref. [9]). Thus, it is natural to assume that the series for higher-order exponents Δ_{nl} also have finite

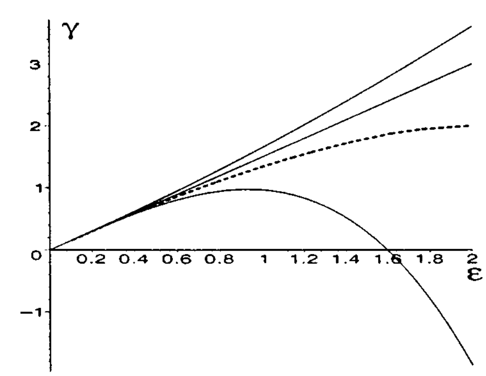

FIG. 3. The exponent γ for $d=2$ vs ε: the $O(\varepsilon^2)$, $O(\varepsilon)$, and $O(\varepsilon^3)$ approximations (from above to below). Dashed line: numerical solution by Refs. [7].

radii of convergence with the behavior similar to that of ε_-. Therefore, in order to obtain reasonable predictions for finite values of ε, one should augment plain ε expansions by the information about the location and character of the singularities. Such information can be extracted from the asymptotical behavior of the coefficients $\Delta_{nl}^{(k)}$ in Eq. (10) at large k. To our knowledge, this problem has never been studied for dynamical models like (1)–(3); the instanton analysis developed in Refs. [15] has mostly been concentrated on the behavior of the exponents in the limit $n \to \infty$. One can hope that the implementation of the instanton calculus within the RG framework will give the solution of this important problem.

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